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Regular and singular components of periodic flows in the fluid interior☆

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Abstract

The structure of infinitesimal periodic motions in the interior of a rotating compressible fluid which has been stratified using salt is analyzed taking account of dissipation effects. In the general case, the system of fundamental equations of motion belongs to the class of singularly perturbed equations, the solutions of that consist of functions which are regular and singular with respect to the dissipative coefficients that describe both propagating hybrid waves as well as several types of accompanying singular components including boundary layers. The thicknesses of the singular components are determined by the kinematic viscosity, the diffusion coefficient of the salt and the characteristic frequencies of the problem. In the model of a barotropic or homogeneous fluid, the singular components of spatial periodic flows combine together, which is indicative of degeneracy of the system of equations. Taking account of the full set of components, which are regular and singular with respect to the dissipative characteristics, enables one to construct exact solutions of problems of the generation and non-linear interaction of waves.

The existence of characteristic types of wave: gyroscopic, internal and acoustic, the general properties of which, as a rule, are analyzed separately in each of the frequency ranges in the ideal fluid approximation[1] is associated with each of the physical factors, rotation, stratification and the compressibility of the fluid. However, elementary waves do not completely reflect the properties of periodic motions when all of the above factors act simultaneously, and hybrid waves exist with complex dispersion laws. Furthermore, dissipative factors, caused by the action of viscosity, thermal conductivity and diffusion, affect the properties of periodic flows.

The effect of thermal conductivity and diffusion on the properties of waves is traditionally neglected and the action of viscosity is taken into account by the introduction of factors which ensure damping.[2] A more detailed analysis shows that taking account of viscosity leads to an increase in the order and a change in the type of equations of motion. The dispersion equations corresponding to them have solutions, both regular and singular with respect to the viscosity, which describe different types of motions.[3] The large-scale weakly attenuating waves correspond to the regular solutions, and flows of the boundary-layer type, the motions in which rapidly decay in some directions and slowly in others, correspond to the singular solutions. Among these are, strictly, boundary layers both on solid boundaries[4] as well as on the free surface of the fluid.[5]

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In experiments, the structural analogues of boundary layers, that is, extended isolated layers with tapered edges that are not in contact with the boundaries of the flows, have been observed in the interior of a stratified fluid in the domains of the intersection of beams of periodic internal waves[6] and in the field of the attached waves behind a horizontal cylinder.[7] Taking full account of the singular components of a flow enables exact solutions to be found of the equations of motion, satisfying the boundary conditions, in the linear[8] and weakly non-linear approximation in which the direct interaction of all of the structural components of a flow are allowed for.[9,10] A classification of periodic flows in a viscous inhomogeneous fluid has been given in Ref. 3. The influence of the effects of thermal conductivity and diffusion on the general properties of the flow, which lead to an increase in the order of the system of equations and to a complication of the analysis, has not been previously studied.

The aim of this paper is to give a the complete classification of the regular and singular components of three-dimensional infinitesimal periodic flows in the interior of a continuously stratified and uniformly rotating fluid on the basis of a fundamental system of equations which include the diffusion equation.

1. The system of equations of motion and boundary conditions

A periodic flow of a heavy compressible fluid on the surface of a solid sphere which rotates at a constant angular velocity Ω is considered. The fluid density ρ depends on the specific entropy s, the pressure p and the mass concentration S of the salt dissolved in it. The unperturbed density distribution is assumed to be an exponentially decreasing distribution with the height z: $\rho_0(z) = \rho_{00}\exp(-z/\Lambda)$, Λ is the buoyancy scale and ρ_{00} is a constant quantity.

The fluid motion is described by equation of state, conservation of mass, diffusion of the salt and momentum transfer[1]

$$\begin{gathered}\rho = \rho(s, p, S), \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \frac{\partial S}{\partial t} + \nabla\cdot(S\mathbf{v}) - \kappa\Delta S = 0 \\ \frac{\partial(\rho\upsilon^i)}{\partial t} + \nabla_j\Pi^{ij} = \rho g^i + 2\rho\varepsilon^{ijk}\upsilon_j\Omega_k\end{gathered} \tag{1.1}$$

where $\mathbf{v}$ is the velocity vector with components υ_i, κ is the diffusion coefficient of the salt, $\Pi^{ij} = \rho\upsilon^i\upsilon^j + p\delta^{ij} - \sigma^{ij}$ are the components of the momentum flow density tensor, δ^{ij} are the components of the fundamental metric tensor, σ^{ij} are the components of the symmetric viscous stress tensor, ε^{ijk} are the components of a unit antisymmetric tensor and g^i are the components of the vector $\mathbf{g}$ of the acceleration due to gravity.

The no-slip condition and the condition of no salt flux across the solid impermeable boundaries Σ bounding the fluid are satisfied

$$\mathbf{v}|_\Sigma = (S\mathbf{v} - \kappa\nabla S)\cdot\mathbf{n}|_\Sigma = 0 \tag{1.2}$$

where $\boldsymbol{n}$ is a normal to the surface Σ.

Only spatial periodic motions with a constant frequency ω are considered, for which the fluid, that is inhomogeneous with respect to its density, remains isothermal. The fluid motions are assumed to be of low intensity, which enables us to linearize the equations of motion, and the stratification is assumed to be weak. The criteria for the applicability of these approximations will be determined below.

2. The dispersion relation

We introduce the Cartesian system of coordinates (x, y, z), the z axis of which is directed towards the zenith and the direction of the x and y axes are chosen taking account of the condition of the equality of the projections of the angular velocity of rotation Ω onto them. In the linear approximation, the system of equations takes the form[3]

$$\frac{1}{c^2}\frac{\partial\bar{p}}{\partial t}-\frac{wg}{c^2}+\nabla\cdot\upsilon+\kappa\Delta\bar{S}=0,\quad \frac{\partial\bar{\rho}}{\partial t}-\frac{w}{\Lambda}+\nabla\cdot\upsilon=0,\quad \frac{\partial\bar{S}}{\partial t}-\frac{w}{\Lambda}-\kappa\Delta\bar{S}=0$$

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\partial\bar{p}}{\partial x}+2\Omega\left(\upsilon\sin\varphi-\frac{1}{\sqrt{2}}w\cos\varphi\right)+\nu\Delta u+\left(\mu+\frac{\nu}{3}\right)\frac{\partial}{\partial x}\nabla\cdot\upsilon\\
\frac{\partial\upsilon}{\partial t} &= -\frac{\partial\bar{p}}{\partial y}+2\Omega\left(\frac{1}{\sqrt{2}}w\cos\varphi-u\sin\varphi\right)+\nu\Delta\upsilon+\left(\mu+\frac{\nu}{3}\right)\frac{\partial}{\partial y}\nabla\cdot\upsilon\\
\frac{\partial w}{\partial t} &= -\frac{\partial\bar{p}}{\partial z}+\sqrt{2}\Omega(u-\upsilon)\cos\varphi+\upsilon\Delta w+\left(\mu+\frac{\nu}{3}\right)\frac{\partial}{\partial z}\nabla\cdot\upsilon-\bar{\rho}g
\end{aligned}\tag{2.1}$$

Here, u, υ, w are the components of the velocity vector $\mathbf{v}$, $\bar{\rho}$ and $\bar{p}$ are the perturbations of the density and pressure, normalized to the quantity $\rho_0(z)$, $\bar{S}$ is the perturbation of the salinity which is normalized to the initial stratified distribution $S_0(z)$, φ is the height of the point of observation, and ν and μ are the first and second kinematic viscosities.

The first equation of system (2.1) is the differential form of writing the first equation of system (1.1) (the equations of state).

The particular solutions of system (2.1) describing motions which are periodic in time with a fixed real frequency ω and wave vector $\mathbf{k}=(k_x, k_y, k_z)$, the components of which, in the general case, take complex values, are specified in the form

$$\mathbf{v}=\mathbf{v}_0\tau(\mathbf{r},t),\quad \bar{p}=p_0\tau(\mathbf{r},t),\quad \bar{\rho}=\rho_0\tau(\mathbf{r},t),\quad \tau(\mathbf{r},t)=\exp(i(\mathbf{kr}-\omega t))\tag{2.2}$$

According to representations (2.2), the small intensity approximation of motions implies that the condition $|\mathbf{k}||\mathbf{v}|\ll\omega$ is satisfied and the weak-stratification approximation holds when $|\mathbf{k}|\boldsymbol{\Lambda}\gg 1$.

Substitution of expressions (2.2) into system (2.1) generates a system of linear algebraic equations in the amplitudes u_0, υ_0, w_0, P_0, ρ_0. The dispersion equation (DE)

$$\begin{aligned}
&D_\kappa(k)\{\omega D_\nu(k)[\omega D_\nu(k)(\tilde{D}_\nu(k)-N^2)-2\sqrt{2}\omega\Omega g(k_y-k_x)\cos\varphi-\\
&-iD_\nu(k)N^2(\mu+\nu/3)k_\perp^2]+4\omega\Omega^2[N^2\sin^2\varphi-\omega(D_\nu(k)+i(\mu+\nu/3)f^2(k))]+\\
&+c^2[D_\nu(k)(N_c^2k_\perp^2-\omega k^2D_\nu(k))+4\omega\Omega^2f^2(k)]\}+\\
&+\kappa c^2k^2\Lambda^{-1}[\omega k_zD_\nu^2(k)-4\omega\Omega^2f(k)\sin\varphi-iD_\nu(k)(gk_\perp^2+\sqrt{2}\omega\Omega(k_y-k_x)\cos\varphi)]=0
\end{aligned}\tag{2.3}$$

follows from the condition for the non-trivial solvability of this system, that is, of the equality of its determinant to zero.

Here,

$$\begin{aligned}
&D_\kappa(k)=\omega+i\kappa k^2,\quad D_\nu(k)=\omega+i\nu k^2,\quad \tilde{D}_\nu(k)=\omega+i\tilde{\nu}k^2,\quad \tilde{\nu}=4\nu/3+\mu,\\
&k^2=k_x^2+k_y^2+k_z^2,\quad k_\perp^2=k_x^2+k_y^2,\quad N^2=g/\Lambda,\quad N_c^2=N^2-g^2/c^2,\\
&f(k)=k_z\sin\varphi+(k_x+k_y)\cos\varphi/\sqrt{2}
\end{aligned}$$

It gives a relation between the three components of the wave vector at a specified frequency ω. The action of gravitational forces separates out one of the components of the wave vector and, here, the dependence $k_z(k_x, k_y)$[2] is conventionally understood to be the solution of the DE. The substitution of this transforms DE (2.3) into an identity for the specified k_x, k_y.

The DE is an eight order polynomial in the components k_z and, in the general case, it has a set of permissible solutions $k_{zj}(k_x, k_y)$ to each of which its own characteristic type of periodic motion corresponds.

It should be especially noted that, in DE (2.3), the small coefficient $\nu^2\tilde{\nu}\kappa$ is present accompanying the leading term $k^8=(k_x^2+k_y^2+k_z^2)^4$, that is, with respect to the wave-number k, the equation belongs to the class of singularly perturbed equations.[11]

The complete solution of the linearized system of equations of motion (2.1) is sought in the form of a superposition of the particular solutions (2.2)

$$A = \sum_j \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a_j(k_x, k_y)\exp(i(k_{zj}(k_x, k_y)z + k_x x + k_y y - \omega t))dk_x dk_y \tag{2.4}$$

The symbols A and a_j refer to each physical quantity (velocity, pressure or density) being considered, and its spectral representation, where the coefficients $a_j(k_x, k_y)$ are determined from the boundary conditions.[8]

It is important to emphasize that the summation in equality (2.4) must be carried out over all roots of DE (2.3), to which physically realizable solutions correspond.[12] The set of such solutions is specified by the boundary conditions of the problem or the condition that all of the perturbations tend to zero at infinity (the radiation condition).

When all of the kinetic coefficients vanish, the DE becomes a second-order equation and, consequently, two of its roots are regular with respect to the kinetic coefficients, that is, they are representable in the form of series in non-negative, not necessarily integral, powers of the kinetic coefficients. The solutions corresponding to them uniformly pass into the solutions of the DE for an ideal fluid and describe the propagation of perturbation waves. Three types of motions, the properties of which are substantially different from wave motions, correspond to the six other solutions, which are singular with respect to the viscosity and the diffusion coefficient of the salt. These solutions are expanded in series in integral and fractional negative powers of the kinetic coefficients.

3. The condition for the existence of propagating waves

An analysis of the properties of the solutions of DE (2.3) is simplified in the spherical system of coordinates (k, ψ, θ) in the space of the wavenumbers (k_z, k_y, k_z) introduced by the relations

$$k_x = k\sin\theta\cos\psi, \quad k_y = k\sin\theta\sin\psi, \quad k_z = k\cos\theta$$

The solutions which are regular with respect to the kinetic coefficients $k_z^{(r)}$ are represented in the form of a power series in the three variables ν, μ and κ

$$k_z^{(r)} = k_0 + \sum_{i,j,k=0}^{\infty} b_{ijk}\kappa^i\nu^j\mu^k \tag{3.1}$$

The zeroth term of the expansion, which is the solution k_0 of DE (2.3) neglecting all dissipative effects, is

$$\begin{aligned} k_0 &= (\beta \pm \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha \\ \alpha &= c^2(N_c^2\sin^2\theta - \omega^2 + 4\Omega^2F^2), \quad \beta = 2\sqrt{2}\omega\Omega g(\sin\psi - \cos\psi)\sin\theta\cos\varphi \\ \gamma &= \omega^2(\omega^2 - N^2) + 4\Omega^2(N^2\sin^2\theta - \omega^2) \\ F &= \cos\theta\sin\varphi + ((\sin\psi + \cos\psi)\sin\theta\cos\varphi)/\sqrt{2} \end{aligned} \tag{3.2}$$

It can be seen, in particular, that the ranges of existence of the propagating waves corresponding to DE (2.3) and characterized by the real frequency ω is determined by the condition $\beta^2 - 4\alpha\gamma \geq 0$, which shows that the boundaries of the frequency ranges for the existence of propagating waves in the fluid interior depends on many factors: the frequencies of the wave ω, the rotation Ω and the buoyancy N, the compressibility of the medium and the geometry of the problem. The frequency boundaries of the ranges of existence of the acoustic ($\Omega = N = 0$) and internal ($\Omega = 0$, $N_c = N$) branches of the periodic motions in stratified compressible media are identical to those established earlier in Ref. 13.

Within the range of existence of inertial-gravitational waves in the ideal fluid approximation, the graphical form of the existence is shown in Fig. 1 in the coordinates φ (patitude) and ω (frequency). The maximum frequency of the range of existence of such waves depends on the buoyancy and rotation frequencies $\omega_{\max} = \sqrt{N^2 + 4\Omega^2}$, where the difference between their maximum and minimum values determines the width of the frequency range of existence of waves at the poles.

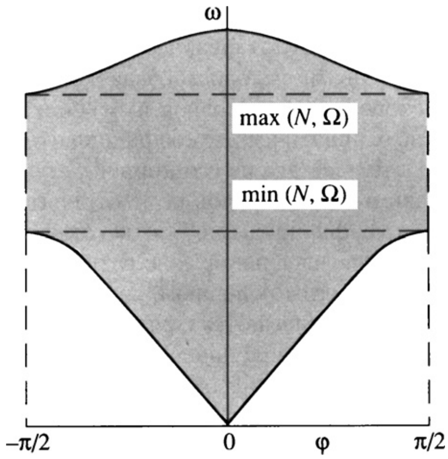

Fig. 1.

Successive application of perturbation theory in the search for the regular solutions of DE (2.3) shows that the corrections of the first approximation to solutions (3.2) are given by the relations

$$k_1 = -\frac{H(k_0)}{\omega \partial E(k_z)/\partial k_z|_{k_z = k_0}} \tag{3.3}$$

where

$$\begin{aligned} H(k_0) &= \kappa c^2 k_0^3 \Lambda^{-1}(\omega^2 \cos\theta - 4\Omega^2 F \sin\theta) + \\ &+ i[\nu(\omega^2(2\omega^2 - N^2) - 4\Omega^2 N^2 \sin^2\varphi - c^2(\omega^2 + 4\Omega^2 F^2)k_0^2) + \\ &+ (\mu + \nu/3)\omega^2(\omega^2 - N^2 \sin^2\theta - \Omega^2 F^2) - \\ &- \kappa c^2 k_0 \Lambda^{-1}(g k_0 \sin\theta + \sqrt{2}\omega\Omega(\sin\psi - \cos\psi)\sin\theta\cos\varphi)] \end{aligned}$$

and $E(k_z)$ is the left-hand side of DE (2.3) in the non-dissipative approximation.

It follows from the form of the real part of the function $H(k_0)$ that the phase corrections due to dissipative effects depend solely on the diffusion coefficient of the salt since, in the first order approximation, the complete regular solution has the form of the sum of expressions (3.2) and (3.3): $k_z^{(r)} = k_0 + k_1$.

The solutions obtained describe two types of hybrid waves, the properties of which are determined by the effect of the rotation of the Earth, by the compressibility and stratification of the medium, and the effects of viscosity and diffusion.

The properties of the singular solutions $k_z^{(r)}$ of DE (2.3) need to be studied in greater detail. It follows from an analysis of particular problems that boundary flows, which arise on moving contact surfaces in problems of the radiation of waves (the solution of the problem of the generation of internal waves in a viscous fluid[8,9] can be mentioned as an example), correspond to them. Components of flows which are similar in structure are also formed on a reflecting surface in the domain of contact of the incident and reflected waves[14] and during the passage of a wave across a surface of discontinuity in the gradient of the higher derivatives of the density.[15]

To illustrate the properties of the singular components of a flow, the structure of periodic boundary layers on a plane, the orientation of the normal to which is characterized by the angles ψ and θ introduced earlier will be analyzed. Here, in order to shorten the account, the treatment is carried out in a local coordinates system of coupled with the surface being analyzed, one axis of which is directed along the normal to the surface while the other two axes are parallel to the plane. In this case, the dependence of the component k_n of the wave vector normal to the plane and its components parallel to the plane is understood to be the solution of the DE. The equation determining the leading terms of the singular solutions in the system of coordinates which has been introduced has the form

$$\varepsilon\nu^3 k_n^6 - i\omega\nu^2[1-\varepsilon(M_c^2 - M^2 - 2)]k_n^4 + \\ + \omega^2\nu[M_c^2 - 2 + \varepsilon(M_c^2 - M^2 - 1 + 4\Omega^2F^2/\omega^2)]k_n^2 - i\omega^3(M_c^2 - 1 + 4\Omega^2F^2/\omega^2) = 0 \\ M^2 = N^2\sin^2\theta/\omega^2, \quad M_c^2 = N_c^2\sin^2\theta/\omega^2, \quad \varepsilon = \kappa/\nu \tag{3.4}$$

(in real liquids, the Schmidt number ν/κ is, as a rule, large and, consequently, $\varepsilon \ll 1$).

In Eq. (3.4) there are no components of the wave vector parallel to the bounding surface. Hence, only those transverse solenoidal motions exist close to the solid surface in which the velocity of the particles parallel to this surface also only changes in a normal direction.

Equation (3.4) has three solution in the variable k_n^2:

$$k_{n\kappa}^2 = \frac{i\omega}{\kappa}(1+\varepsilon M^2 - \varepsilon^2 M^2(M_c^2-1)), \quad k_{n\nu\pm}^2 = -\frac{i}{\nu}\left(\omega_\pm - \omega + \varepsilon M^2\frac{M_c^2-1}{M_c^2 - 2\omega_\pm/\omega}\right) \tag{3.5}$$

Here,

$$\omega_\pm = \frac{\omega M_c^2}{2}\left(1 \pm \sqrt{1+\frac{16\Omega^2F^2}{\omega^2 M_c^4}}\right) \tag{3.6}$$

The difference in the structures of the characteristic frequencies ω_+ and ω_- and in the properties of the boundary layers shows up most clearly in the case of small values of the angular velocity of rotation Ω.

A periodic density boundary layer corresponds to the solution $k_{n\kappa}^2$, the characteristic scale of which is equal to

$$\delta_\kappa \approx \sqrt{2\kappa/\omega} \tag{3.7}$$

Two viscous periodic boundary layers correspond to the two other solutions $k_{n\nu\pm}^2$ with the characteristic scales

$$\delta_{\nu\pm} \approx \sqrt{2\nu/|\omega_\pm - \omega|} \tag{3.8}$$

All of the above mentioned motions exist simultaneously and their scales depend on the parameter ε.

It follows from relations (3.6) and (3.8) that, in the case of small rotational frequencies $\Omega \ll \omega M_c^2/(4F)$, the viscous boundary layers are substantially different in their nature. Since, in this case,

$$\omega_+ \approx \omega M_c^2, \quad \omega_- \approx -\frac{4\Omega^2F^2}{\omega M_c^2}$$

we have

$$\delta_{\nu+} \approx \sqrt{\frac{2\nu}{\omega}}|M_c^2 - 1|^{-1/2}, \quad \delta_{\nu-} \approx \sqrt{\frac{2\nu}{\omega}}\left(1+\frac{4\Omega^2F^2}{\omega^2M_c^4}\right)^{-1/2} < \sqrt{\frac{2\nu}{\omega}} \tag{3.9}$$

The angular velocity of rotation of the sphere in which the fluid is located does not enter into the definition of the thickness of one of these layers ($\delta_{\nu+}$), which is indicative of its acoustic-gravitational origin. The other layer, with a thickness $\delta_{\nu-}$, has a more complex structure which is influenced by all of the effects being analyzed (rotation, stratification and compressibility).

In the case of high frequencies of the oscillations ω when the conditions

$$M_c^2 \ll 1, \quad 2\Omega F \ll \omega M_c^2$$

are satisfied, the thicknesses of viscous boundary layers tend towards the Stokes thickness of a periodic boundary layer

$$\delta_{\nu+} \approx \delta_{\nu-} \approx \sqrt{2\nu/\omega} \tag{3.10}$$

Equations (2.1) and DE (2.3) serve as a basis for investigating the properties of simpler types of harmonic motions when, in real dissipative media, only some of the factors being analyzed shows up or when the dimension of the motion is less than three.

Since it is usual in real media that $\kappa \ll \nu$ $(\varepsilon \ll 1)$, according to relations (3.5), density and velocity boundary layers split, that is, the properties of each of them depends on a single dissipative factor

$$k_{n\kappa}^2 \approx i\omega/\kappa, \quad k_{n\nu\pm}^2 \approx -i(\omega_\pm - \omega)/\nu$$

The effect of diffusion of the salt shows up most obviously in the existence of a thin (when $\varepsilon \ll 1$) density boundary layer. Its action on the propagation of waves in the situation being investigated is far smaller than the role of viscosity, which is confirmed by the form of the first order correction with respect to the kinetic coefficients (3.5). The effect of viscosity on viscous boundary layers is also small. Henceforth, all of the limiting cases, which do not touch upon the existence of a density boundary layer, are considered in the diffusion-free approximation.

4. Characteristic types of periodic flows

When the effects of rotation are neglected $(\Omega = 0)$, it follows from expressions (3.5) that propagating *gravitational waves* exist in two frequency ranges $\omega \le N_c$ and $\omega \ge N$. At lower frequencies $(\omega \ll N_c)$, they display the properties of internal gravitational waves and, at high frequencies $(\omega \gg N)$, approach, as regards their character, acoustic waves in an isotropic medium. In the latter case, two gravitational-acoustic boundary layers with the characteristic scales

$$\begin{aligned} \delta_{b-} &= \delta N\sqrt{2/\sin\theta_\omega} \\ \delta_{b+} &= \delta_N\sqrt{\frac{2\sin\theta_\omega}{\left|(1-g\Lambda/c^2)\sin^2\theta - \sin^2\theta_\omega\right|}} \approx \delta_N\sqrt{\frac{2\sin\theta_\omega}{\left|\sin^2\theta - \sin^2\theta_\omega\right|}} \\ \delta_N &= \sqrt{\nu/N}, \quad \theta_\omega = \arcsin(\omega/N) \end{aligned} \tag{4.1}$$

are formed on the rigid boundaries in the medium. The first of them is similar to the periodic Stokes flow of a homogeneous fluid, and the second, with parameters which depend both on the stratification (N) as well as on the compressibility (c), is spherical in the case of stratified media. Direct calculations of the divergence of the velocity show that the fluid in the boundary-layer flow behaves as an incompressible fluid in spite of the fact that the existence of such a periodic flow is partially due to the effects of compressibility.

Inertial-gravitational waves in stratified rotating incompressible media are also supplemented by two types of boundary layer with the scales

$$\delta_{b\pm} = \delta_N\sqrt{\frac{2}{|\omega_\pm - \omega^*|}}, \quad \omega_\pm = \frac{\sin^2\theta}{2\omega^*}\left(1 \pm \sqrt{1 + \frac{16\Omega^2F^2}{\omega^2M^4}}\right), \quad \omega^* = \frac{\omega}{N} \tag{4.2}$$

The frequency range for the existence of inertial-gravitational waves is bounded by the values

$$\omega_{1,2} = \left\{\frac{1}{2}\left(N^2 + 4\Omega^2 \pm \sqrt{(N^2+4\Omega^2)^2 - 16N^2\Omega^2\sin^2\varphi}\right)\right\}^{1/2}$$

and depends on the patitude of the point of observation.

Inertial-acoustic waves in a non-stratified fluid $(N = 0)$ are supplemented by boundary layers with the scales

$$\delta_{b\pm} = \delta_N\sqrt{\frac{\nu}{\Omega|F \pm \cos\theta_\Omega|}}; \quad \theta_\Omega = \arccos\frac{\omega}{2\Omega} \tag{4.3}$$

where θ_Ω is the angle to the horizon at which the waves propagate. Periodic perturbations display the properties of inertial waves when $\omega \ll \Omega$ and acoustic waves in the opposite case.

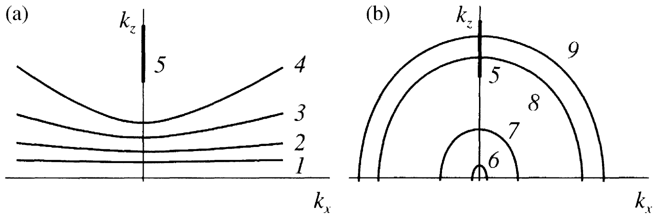

Fig. 2.

When viscosity is neglected, the DE for such waves takes the form

$$k_x^2 + k_y^2 + \frac{\omega^2 - 4\Omega^2}{\omega^2}\left(k_z^2 - \frac{\omega^2}{c^2}\right) = 0 \tag{4.4}$$

and becomes the classical DE for inertial waves when $\omega < 2\Omega$ and for acoustic waves in the high frequency limit $\omega \gg 2\Omega$.[1] The dispersion surfaces have the form of hyperboloids of revolution in the first case and ellipsoids of revolution in the second.

The sections of the dispersion surfaces with the plane $k_y = 0$ are symmetric in the case of reflections with respect to the axis $k_x = 0$. Their form in the upper half-plane (k_x, k_z) is shown in Fig. 2 in the limit of low (*a*) and high (*b*) frequencies; the frequency of the wave increases as the number of the dispersion curve becomes larger. It follows from the form of DE (4.4) that, when the frequency is reduced, the dispersion curves straighten out and approach the k_x axis for the case when $\omega \ll 2\Omega$. The dispersion curves for the inertial waves when $\omega < 2\Omega$ degenerate into the segments marked with the number 5 when $\omega = 2\Omega$ (Fig. 2, *a*) and the inertial waves (like the internal waves with a buoyancy frequency $\omega = N$) are transformed into oscillations with zero group velocity. The dispersion curves, which have an elliptic form when $\omega > 2\Omega$ (Fig. 2, *b*), approximate to a circle as the frequency increases $\omega \ll 2\Omega$.

Attenuating *acoustic waves* in a homogeneous quiescent medium ($N = \Omega = 0$) are characterized by the DE

$$\omega^2 = k^2(c^2 - i\omega(4\nu/3 + \mu)) \tag{4.5}$$

the real part of which determines the dispersion of sound in an ideal fluid while the imaginary part, describing the dissipation, depends both on the first as well as the second viscosity. The DE (4.5) only describes a single boundary layer with a thickness $\delta_b = \sqrt{2\tilde{\nu}/\omega}$, $\tilde{\nu} = 4\nu/3 + \mu$ which, in fact, is a doubly degenerate layer. It is formed by the merging of two differing gravitational-acoustic boundary layers, the thicknesses of which are given by relation (4.1). When $N \to 0$, the boundary layer become identical and the order of the DE simultaneously decreases. The perturbations within an acoustic boundary layer are transverse with zero divergence of the velocity, that is, as in the case of gravitational-acoustic waves, the fluid in it behaves as an incompressible fluid.

Internal gravitational waves in a non-rotating ($\Omega = 0$) incompressible medium coexist with a viscous wave boundary layer with a scale $\delta_b = \sqrt{2\nu/\omega}$. This is an analogue of a periodic Stokes flow on an oscillating surface in a homogeneous fluid and an internal wave boundary layer, the characteristic scale of which

$$\delta_w = \sqrt{2\nu\omega/|N^2\sin^2\theta - \omega^2|}$$

depends explicitly on the buoyancy frequency N. The thicknesses of the boundary layers remain finite for any positions of the surface bounding the fluid.[14] Hence, the paradox of "critical angles" (of the infinite compression of a beam when the radiating or reflecting plane coincides with the propagation ray of a wave[16]), which is a consequence of the ideal fluid approximation or the incorrect application of asymptotic methods in calculating the wave pattern in dissipative media, is resolved by the correct use of the viscous theory fluid.

Inertial waves in a homogeneous ($N = 0$) incompressible medium are supplemented with two gyroscopic wave boundary layers, the scales of which are given by the expressions

$$\delta_{g\pm} = \sqrt{\nu/|\omega/2 \pm \Omega\cos\theta|}$$

A more detailed analysis shows that singularities in the expression for the thickness of the boundary layers do not arise in the exact solution for any positions of the radiating surface, even in the critical case when its inclination coincides with the direction of propagation of an inertial wave as in the case of internal waves.[14]

The dispersion equation for periodic perturbations of a *homogeneous incompressible fluid* ($N = \Omega = 0$) has the simplest form

$$k^2 D_\nu^2(k) = 0 \tag{4.6}$$

The first factor characterizes pseudowaves $k^2 = 0$ in which all three components of the wave vector $\mathbf{k}$ are non-zero. Such motions possess the property $\mathbf{v} = p\mathbf{k}/\rho$ and taking account of incompressibility leads to $\mathbf{k}$ and $\mathbf{v}$ having complex values.

When spatial motions about a rigid oscillating boundary are considered, a doubly degenerate viscous wave boundary layer arises with a thickness $\delta_b = \sqrt{2\nu/\omega}$, consisting of two periodic Stokes waves.

5. Conclusion

The family of infinitesimal periodic flows in a fluid in both the two-dimensional and three-dimensional cases includes large-scale waves and two families of singular components (of the type of boundary layers: two viscous boundary layers and one density boundary layer), coexisting with them, with different characteristic wavelength scales and transverse dimensions of the layers. For generation by compact sources, the regular and singular components of the periodic flows are formed and disappear simultaneously in spite of the differences in their characteristic spatial scales. Each of the forms of the flows ensures the transfer of energy and matter in the domains of its own localization. By virtue of the non-linearity of the equations, all the structural components, both the regular and the singular, directly interact with one another and can generate new components of the flows: both large-scale waves and vortices as well as finely structured laminae which considerably extends the number of scenarios for the subsequent transformation of the pattern of flows.[9,10]

The main energy of the motions is included in the regular components, while vorticity and dissipation are included in the singular components. Hence, taking account of the singular components which have a substantial effect on the level and transfer of vorticity and, also, on the transfer of matter in the fluid, automatically excludes the condition for the motions to be potential motions, which is frequently used in the theory of waves. In limiting cases (the approximation of a barotropic or homogeneous fluid) the singular spatial components become identical and merge, which leads to degeneration of the problem.

The inclusion of all the regular and singular components of periodic flows in the analysis enables us to construct, in the linear approximation, exact solutions of problems of the generation and reflection of internal waves, which do not contain empirical coefficients.[8,9] Taking account of the dependence of the density of the medium on temperature leads to an enlargement of the number of singular components, which, in this case, can be both completely as well as partially separated.[17]

In the general case, the solutions of problems within the framework of models of stratified, rotating, compressible, viscous media, taking account of diffusion and thermal diffusivity, assume a uniform approach to the model of a homogeneous viscous fluid or to a homogeneous ideal isothermal fluid. However, in this case, a number of the singular components of the solutions is lost.

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